

Additive bases with coefficients of newforms

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Abstract

Let $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ be a normalized Hecke eigenform in $S_{2k}^{\text{new}}(\Gamma_0(N))$ with integer Fourier coefficients. We prove that there exists a constant $C(f) > 0$ such that any integer is a sum of at most $C(f)$ coefficients $a(n)$. It holds $C(f) \ll_{\varepsilon, k} N^{\frac{6k-3}{16} + \varepsilon}$.

Key words: newform; Fourier coefficients; additive basis

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1 Introduction

The set $\{\tau(n) \mid n \geq 1\}$ of values of Ramanujan's function is an additive basis of the integers: any integer Z can be written as

$$Z = \sum_{j=1}^{74000} \tau(n_j).$$

See [2], [11]. Here we prove a similar property for the Fourier coefficients of normalized Hecke eigenforms. For integers $k \geq 1$, $N \geq 1$ denote by $S_{2k}^{\text{new}}(\Gamma_0(N))$ the space of newforms of weight $2k$ on $\Gamma_0(N)$.

Theorem 1. *Let $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ be a normalized Hecke eigenform in $S_{2k}^{\text{new}}(\Gamma_0(N))$ with integer Fourier coefficients. There exists a constant $C(f) > 0$ such that any integer Z is a sum*

$$Z = \sum_{j=1}^{\ell} a(n_j) \tag{1}$$

for some $\ell \leq C(f)$ and integers $n_j \ll |Z|^{\frac{2}{2k-1}} + 1$. It holds

$$C(f) \ll_{\varepsilon, k} N^{\frac{6k-3}{16} + \varepsilon}.$$

Our method follows the idea of [2] to connect the solubility of (1) with the Waring–Goldbach problem. We use results of Ram Murty on oscillations of Fourier coefficients of newforms, of Matomäki on signs of the coefficients, and of Hua on the Waring–Goldbach problem, which are stated in the second section. The third section contains the proof of the theorem.

2 Lemmas

We apply the following facts.

Lemma 2 (Ram Murty). *Let $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ be a normalized Hecke eigenform in $S_{2k}^{\text{new}}(\Gamma_0(N))$. For any $\varepsilon > 0$ we have*

$$|a(p)| > (\sqrt{2} - \varepsilon)p^{\frac{2k-1}{2}}$$

for a positive density of primes p .

Proof. This was proved in [9, Corollary 2] for forms on the full modular group, but the statement is true also for forms on $\Gamma_0(N)$ [10, Chapter 4, Theorem 8.6 (ii) with $m = 1$, page 89]. \square

Lemma 3. *Let $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ be a normalized Hecke eigenform in $S_{2k}^{\text{new}}(\Gamma_0(N))$. Let n_f be the smallest integer such that $a(n_f) < 0$ and $(n_f, N) = 1$. Then*

$$n_f \ll (4k^2N)^{\frac{3}{8}},$$

where the implied constant is absolute.

Proof. See [8, Theorem 1]. See also [5], [4], [6]. \square

Let $k \geq 1$ be an integer, p a prime number, $\theta \geq 0$ the integer with $p^\theta \mid k$ and $p^{\theta+1} \nmid k$,

$$\gamma = \begin{cases} \theta + 2, & \text{if } p = 2 \text{ and } \theta > 0, \\ \theta + 1, & \text{otherwise,} \end{cases} \quad (2)$$

$$K = \prod_{(p-1) \mid k} p^\gamma. \quad (3)$$

Lemma 4 (Hua). *Let $k \geq 1$ be an integer and K as in (3). If*

$$s \geq \begin{cases} 2^k, & \text{if } k \leq 10 \\ 2k^2(2 \log k + \log \log k + \frac{5}{2}), & \text{if } k > 10, \end{cases}$$

then for any $Z \equiv s \pmod{K}$ the number $I_s(Z)$ of solutions of the equation

$$p_1^k + \cdots + p_s^k = Z, \quad \text{for primes } p_1, \dots, p_s, \quad (4)$$

satisfies the following asymptotic formula

$$I_s(Z) = \mathfrak{G}(Z) \frac{\Gamma^s(1/k)}{\Gamma(s/k)} \frac{Z^{\frac{s}{k}-1}}{\log^s Z} + O_{k,s} \left(\frac{Z^{\frac{s}{k}-1}}{\log^{s+1} Z} \log \log Z \right), \quad (5)$$

where $\mathfrak{G}(Z)$ is the singular series

$$\mathfrak{G}(Z) = \sum_{q=1}^{\infty} \left(\sum_{(h,q)=1} \left(\sum_{(l,q)=1} e^{2\pi i h \frac{l^k}{q}} \right)^s e^{2\pi i \frac{-h}{q} Z} \right),$$

which is absolutely convergent and there exist positive constants A, B independent of Z such that

$$0 < A \leq \mathfrak{G}(Z) < B.$$

Proof. See [3, Theorem 11 and Theorem 12, pages 78 and 100 respectively]. \square

Kumchev and Wooley proved in [7, Theorem 1] that equation (4) has solution if k is large and $s \geq (4k - 2) \log k + k - 7$.

3 Proof of the theorem

We denote by \mathcal{P} the set of prime numbers which do not divide N and $\mathcal{P}(X) = \mathcal{P} \cap [1, X]$. By Lemma 3 let $n_f > 1$ be the smallest integer such that $a(n_f) < 0$ and $(n_f, N) = 1$. Let M be a large parameter and set

$$\mathcal{P}_0(M) = \mathcal{P} \cap (n_f, M].$$

We will say that $D_M \subset \mathcal{P}_0(M)$ is an admissible subset of $\mathcal{P}_0(M)$ if

$$\sum_{i=1}^k a(p_i) \neq \sum_{i=k+1}^{2k} a(p_i)$$

for any $p_1, \dots, p_{2k} \in D_M$ such that

$$p_1 < \dots < p_k, \quad p_{k+1} < \dots < p_{2k}, \quad (p_1, \dots, p_k) \neq (p_{k+1}, \dots, p_{2k}).$$

We prove that admissible sets exist. The Ramanujan–Petersson conjecture, proved by Deligne [1], states that $|a(p)| \leq 2p^{\frac{2k-1}{2}}$ for any prime number p . Let $\mathcal{P}' \subset \mathcal{P}$ be the set of prime numbers such that

$$p^{\frac{2k-1}{2}} < |a(p)|. \tag{6}$$

From Lemma 2 it follows that there exists a constant $0 < \alpha \leq 1$, which depends on f , such that for T large enough we have

$$\mathcal{P}' \cap [1, T] = \alpha \pi(T) (1 + o(1)), \tag{7}$$

where $\pi(T)$ is the prime counting function. Let $\ell_0 > 10 \log k$ be an integer. For any $1 \leq i \leq 2k$ let

$$A_i := \mathcal{P}' \cap [2^{\ell_0 i}, 2^{\ell_0 i + 1}]. \tag{8}$$

From (7) it follows that

$$|A_i| > \frac{\alpha}{2} \frac{2^{\ell_0 i}}{\log 2^{\ell_0 i}},$$

so we can choose M and ℓ_0 sufficiently large such that $A_i \subset \mathcal{P}_0(M)$ and $|A_i| > 1$ for any $1 \leq i \leq 2k$. Let $p_i \in A_i$, $1 \leq i \leq 2k$. From (6) and the Ramanujan-Petersson-Deligne estimate $|a(p_i)| \leq 2p_i^{\frac{2k-1}{2}}$ it follows that

$$|a(p_1)| < \cdots < |a(p_k)| < |a(p_{k+1})| < \cdots < |a(p_{2k})|. \quad (9)$$

The set $\mathcal{Q} := \{p_1, \dots, p_{2k}\}$ is an admissible subset of $\mathcal{P}_0(M)$. Indeed, let $q_1, \dots, q_{2k} \in \mathcal{Q}$ be such that

$$q_1 < \cdots < q_k, \quad q_{k+1} < \cdots < q_{2k}, \quad (q_1, \dots, q_k) \neq (q_{k+1}, \dots, q_{2k}), \quad (10)$$

and

$$\sum_{i=1}^k a(q_i) = \sum_{i=k+1}^{2k} a(q_i). \quad (11)$$

Let t be the largest index $1 \leq t \leq k$ such that $q_t \neq q_{k+t}$. From (11) it follows that

$$\begin{aligned} \sum_{i=1}^t a(q_i) &= \sum_{i=k+1}^{k+t} a(q_i), \\ a(q_{k+t}) &= \sum_{i=1}^t a(q_i) - \sum_{i=k+1}^{k+t-1} a(q_i). \end{aligned}$$

From (9) and (10) it follows that

$$|a(q_1)| < \cdots < |a(q_t)|, \quad |a(q_{k+1})| < \cdots < |a(q_{k+t})|,$$

hence

$$\begin{aligned} |a(q_{k+t})| &= \left| \sum_{i=1}^t a(q_i) - \sum_{i=k+1}^{k+t-1} a(q_i) \right| \leq \sum_{i=1}^t |a(q_i)| + \sum_{i=k+1}^{k+t-1} |a(q_i)| \\ &\leq k(|a(q_t)| + |a(q_{k+t-1})|). \end{aligned} \quad (12)$$

Without loss of generality we can suppose that $q_t < q_{k+t}$. Let $1 \leq s \leq 2k$ be such that $q_{k+t} = p_s$. It holds that $q_t, q_{k+t-1} \leq p_{s-1}$, and from (6), (8) and the Ramanujan-Petersson-Deligne estimate $|a(p_{s-1})| \leq 2p_{s-1}^{(2k-1)/2}$ it follows that

$$k(|a(q_t)| + |a(q_{k+t-1})|) \leq 2k|a(p_{s-1})| \leq 4k2^{(\ell_0(s-1)+1)\frac{2k-1}{2}},$$

$$|a(q_{k+t})| = |a(p_s)| \geq 2^{\ell_0 s \frac{2k-1}{2}}.$$

From the above estimates and (12) it follows that

$$2^{\ell_0 s(2k-1)/2} \leq |a(q_{k+t})| \leq k(|a(q_t)| + |a(q_{k+t-1})|) \leq 4k2^{(\ell_0(s-1)+1)\frac{2k-1}{2}},$$

hence

$$\ell_0 \leq 1 + \frac{\log 4k}{(2k-1)\log 2},$$

which contradicts the assumption $\ell_0 > 10 \log k$. So \mathcal{Q} is an admissible subset of $\mathcal{P}_0(M)$.

Let $\mathcal{P}'_0(M) \subset \mathcal{P}_0(M)$ be some admissible subset with largest cardinality. We prove that

$$2k \leq |\mathcal{P}'_0(M)| \ll_k M^{\frac{2k-1}{2k}}. \quad (13)$$

Since the admissible subset \mathcal{Q} constructed above has $2k$ elements it follows that

$$2k \leq |\mathcal{P}'_0(M)|.$$

Let

$$S_k := \{a(p_1) + \dots + a(p_k) : p_1 < \dots < p_k, p_i \in \mathcal{P}'_0(M)\}.$$

Given $\lambda \in S_k$, let $T(\lambda)$ be the number of solutions (p_1, \dots, p_k) of the equation

$$a(p_1) + \dots + a(p_k) = \lambda, \quad p_1 < \dots < p_k, \quad p_i \in \mathcal{P}'_0(M).$$

It holds that

$$\sum_{\lambda \in S_k} T(\lambda) \gg_k |\mathcal{P}'_0(M)|^k.$$

The Cauchy–Schwarz inequality implies that

$$|\mathcal{P}'_0(M)|^{2k} \ll_k \left(\sum_{\lambda \in S_k} T(\lambda) \right)^2 \leq |S_k| \sum_{\lambda \in S_k} T^2(\lambda). \quad (14)$$

Note that $\sum_{\lambda \in S_k} T^2(\lambda)$ is the number of solutions of the equation

$$a(p_1) + \dots + a(p_k) = a(p_{k+1}) + \dots + a(p_{2k}), \quad (15)$$

with

$$p_1 < \dots < p_k, \quad p_{k+1} < \dots < p_{2k}, \quad p_i \in \mathcal{P}'_0(M).$$

Since $\mathcal{P}'_0(M)$ is admissible (15) holds only if $(p_1, \dots, p_k) = (p_{k+1}, \dots, p_{2k})$. From this and (14) it follows that

$$|\mathcal{P}'_0(M)|^k \ll_k |S_k|. \quad (16)$$

The estimate $|a(p)| \leq 2p^{\frac{2k-1}{2}}$ implies

$$|S_k| \ll_k M^{\frac{2k-1}{2}},$$

so from (16) we have

$$|\mathcal{P}'_0(M)| \ll_k M^{\frac{2k-1}{2k}}$$

and (13) is proved.

Let $p \in \mathcal{P}_0(M) \setminus \mathcal{P}'_0(M)$. We proceed as in [2, Page 39] to prove that there exist p_1, \dots, p_{2k-1} in $\mathcal{P}'_0(M)$ such that

$$a(p) = \sum_{i=1}^k a(p_i) - \sum_{i=k+1}^{2k-1} a(p_i).$$

Indeed, the maximality of $\mathcal{P}'_0(M)$ implies that there exist

$$q_1, \dots, q_{2k} \in \mathcal{P}'_0(M) \cup \{p\}$$

such that

$$\sum_{i=1}^k a(q_i) = \sum_{i=k+1}^{2k} a(q_i), \quad q_1 < \dots < q_k, \quad q_{k+1} < \dots < q_{2k}, \quad (q_1, \dots, q_k) \neq (q_{k+1}, \dots, q_{2k}). \quad (17)$$

Moreover,

$$p \in \{q_1, \dots, q_{2k}\},$$

and, by (17), p occurs at most twice in the sequece q_1, \dots, q_{2k} . If p occurs twice, then it appears in q_1, \dots, q_k and in q_{k+1}, \dots, q_{2k} , thus

$$\sum_{i=1}^{k-1} a(q'_i) = \sum_{i=k+1}^{2k-1} a(q'_i),$$

for some q'_1, \dots, q'_{2k} in $\mathcal{P}'_0(M)$ with

$$q'_1 < \dots < q'_k, \quad q'_{k+1} < \dots < q'_{2k}, \quad (q'_1, \dots, q'_k) \neq (q'_{k+1}, \dots, q'_{2k}).$$

This is impossible, since $\mathcal{P}'_0(M)$ is admissible with at least $2k$ elements. Therefore, for any $p \in \mathcal{P}_0(M) \setminus \mathcal{P}'_0(M)$ there exist p_1, \dots, p_{2k-1} in $\mathcal{P}'_0(M)$ such that

$$a(p) = \sum_{i=1}^k a(p_i) - \sum_{i=k+1}^{2k-1} a(p_i).$$

Multiplying by $a(p)$ and taking into account that $(p, p_i) = 1$ we get

$$a(p)^2 = \sum_{i=1}^k a(pp_i) - \sum_{i=k+1}^{2k-1} a(pp_i),$$

since the coefficients of f are multiplicative. Subtracting $a(p^2)$ and applying the identity $p^{2k-1} = a(p)^2 - a(p^2)$ which is satisfied by the coefficients of f it follows that

$$p^{2k-1} = \sum_{i=1}^k a(pp_i) - \sum_{i=k+1}^{2k-1} a(pp_i) - a(p^2). \quad (18)$$

Let

$$s_0 \geq \begin{cases} 2^{2k-1}, & \text{if } 2k-1 \leq 10 \\ 2(2k-1)^2(2 \log(2k-1) + \log \log(2k-1) + \frac{5}{2}), & \text{if } 2k-1 > 10. \end{cases}$$

We prove that for Z large there exist $p_1, \dots, p_{s_0} \in \mathcal{P}_0(Z^{1/(2k-1)}) \setminus \mathcal{P}'_0(Z^{1/(2k-1)})$ such that

$$Z = p_1^{2k-1} + \dots + p_{s_0}^{2k-1}.$$

Let $Z_k := Z^{\frac{1}{2k-1}}$ and

$$K = \prod_{p-1|2k-1} p^\gamma,$$

with γ defined as in (2). Since $2k-1$ is odd, the only prime number p with $p-1|2k-1$ is $p=2$, and for $p=2$ we have

$$\theta = 0, \quad \gamma = 1,$$

hence

$$K = 2.$$

By Lemma 4 there exists a positive constant $c_1 = c_1(k)$ such that for any $Z \equiv s_0 \pmod{2}$, with Z large the number of solutions $I_{s_0}(Z)$ of

$$p_1^{2k-1} + \dots + p_{s_0}^{2k-1} = Z \quad (19)$$

with $p_1, \dots, p_{s_0} \in \mathcal{P}_0(Z_k)$, satisfies

$$I_{s_0}(Z) \geq c_1 \frac{Z^{\frac{s_0}{2k-1}-1}}{\log^{s_0} Z}. \quad (20)$$

Now consider equation (19) with at least one $p_i \in \mathcal{P}'_0(Z_k)$ and denote by $I'_{s_0}(Z)$ its number of solutions. $I'_{s_0}(Z)$ should be less than $s_0 I'_{s_0-1}$, where I'_{s_0-1} denotes the number of solutions of the equation

$$p_1^{2k-1} + \dots + p_{s_0-1}^{2k-1} + p_{s_0}^{2k-1} = Z, \quad (21)$$

with $p_1, \dots, p_{s_0-1} \in \mathcal{P}_0(Z_k)$, and $p_{s_0} \in \mathcal{P}'_0(Z_k)$. Note that

$$I'_{s_0-1} = \sum_{p_{s_0} \in \mathcal{P}'_0(Z_k)} I'_{s_0-1}(Z - p_{s_0}^{2k-1}),$$

where $I'_{s_0-1}(Z - p_{s_0}^{2k-1})$ denotes the number of solutions of (21) for p_{s_0} given. Therefore we have

$$I'_{s_0-1} \leq \max_{p_{s_0} \in \mathcal{P}'_0(Z_k)} \{I'_{s_0-1}(Z - p_{s_0}^{2k-1})\} \sum_{p_{s_0} \in \mathcal{P}'_0(Z_k)} 1.$$

Afterwards, for some $p'_{s_0} \in \mathcal{P}'_0(Z_k)$ we get

$$I'_{s_0-1} \leq I'_{s_0-1}(Z - p_{s_0}'^{2k-1}) |\mathcal{P}'_0(Z_k)|. \quad (22)$$

In order to estimate $I'_{s_0-1}(Z - p_{s_0}'^{2k-1})$ we apply Lemma (4) with $s_0 - 1$ variables. Recalling that $Z - p_{s_0}'^{2k-1} > n_f^{2k-1}$, we obtain

$$I'_{s_0-1}(Z - p_{s_0}'^{2k-1}) \ll \frac{(Z - p_{s_0}'^{2k-1})^{\frac{s_0-1}{2k-1}-1}}{\log^{s_0-1}(Z - p_{s_0}'^{2k-1})} \ll \frac{Z^{\frac{s_0-1}{2k-1}-1}}{\log^{s_0-1} Z}. \quad (23)$$

Combining equations (22), (23) and estimate (13) we get

$$I'_{s_0-1} \ll |\mathcal{P}'_0(Z_k)| \frac{Z^{\frac{s_0-1}{2k-1}-1}}{\log^{s_0-1} Z} \ll_k \frac{Z^{\frac{s_0}{2k-1}-1-\frac{1}{2k(2k-1)}}}{\log^{s_0-1} Z}. \quad (24)$$

The number of solutions for (19) with $p_i \in \mathcal{P}_0(Z^{1/(2k-1)}) \setminus \mathcal{P}'_0(Z^{1/(2k-1)})$ is equal to $I_{s_0}(Z) - I'_{s_0}(Z)$. The estimates (20) and (24) imply that

$$I_{s_0}(Z) - I'_{s_0}(Z) \geq I_{s_0}(Z) - s_0 I'_{s_0-1} \gg_k \frac{Z^{\frac{s_0}{2k-1}-1}}{\log^{s_0} Z} \left(1 - \frac{\log Z}{Z^{1/(2k(2k-1))}}\right).$$

Therefore equation (19) is solvable for primes in $\mathcal{P}_0(Z^{1/(2k-1)}) \setminus \mathcal{P}'_0(Z^{1/(2k-1)})$. From this and (15) it follows that any large integer Z with $Z \equiv s_0 \pmod{2}$ has a representation

$$Z = \sum_{i=1}^{ks_0} a(n_i) - \sum_{j=1}^{ks_0} a(n_j)$$

for some integers $n_i, n_j \leq Z^{2/(2k-1)}$ with $(n_f!N, n_i) = (n_f!N, n_j) = 1$. Note that $-Z$ has a similar representation. We also note that any integer Z_0 can be represented as

$$Z_0 = r_0 + Z$$

with $Z \equiv s_0 \pmod{2}$, $0 \leq r_0 < 2$, thus if Z_0 is large then

$$Z_0 = \sum_{i=1}^{ks_0} a(n_i) - \sum_{j=1}^{ks_0} a(n_j) + \underbrace{a(1) + \cdots + a(1)}_{r_0\text{-times}},$$

since $a(1) = 1$. Recall that n_f satisfies $a(n_f) < 0$. Let $C_0 := -a(n_f)$. We have

$$C_0 Z_0 = C_0 \sum_{i=1}^{ks_0} a(n_i) + \sum_{j=1}^{ks_0} a(n_j n_f) + C_0 r_0 a(1)$$

with $0 \leq r_0 < K$. As above, we note that for Z_1 large enough there exist integers Z_0 and $0 \leq r_1 < C_0$ such that

$$Z_1 = C_0 Z_0 + r_1 = C_0 \sum_{i=1}^{ks_0} a(n_i) + \sum_{j=1}^{ks_0} a(n_j n_f) + C_0 r_0 a(1) + r_1 a(1). \quad (25)$$

Therefore, any integer Z with $|Z| \geq T$ can be expressed as in (25) with $r_0 \leq 2$, $r_1 \leq C_0$. The number of summands $a(n)$ in (25) is

$$(C_0 + 1)ks_0 + C_0 r_0 + r_1.$$

For integers Z with $|Z| \leq T$ let n' be such that $2Z < a_{n'}$. It holds that $|Z - a(n')| > T$, so $Z - a(n')$ can be written in the form (25). Hence any integer Z can be written in the form

$$Z = \sum_{j=1}^{\ell} a(n_j),$$

with

$$\ell \leq (C_0 + 1)ks_0 + C_0r_0 + r_1 + 1 \leq (C_0 + 1)ks_0 + 2C_0 + C_0 + 1 =$$

$$= (1 - a(n_f))ks_0 - 3a(n_f) + 1 = -a(n_f)(ks_0 + 3) + ks_0 + 1,$$

since $r_0 \leq 2$ and $r_1 \leq C_0$. The theorem is proved with

$$C(f) := -a(n_f)(ks_0 + 3) + ks_0 + 1.$$

Since s_0 depends only on k we have

$$C(f) \ll_k |a(n_f)|.$$

By the Ramanujan-Petersson-Deligne estimate we have

$$|a(n_f)| \leq d(n_f)n_f^{\frac{2k-1}{2}},$$

where $d(\cdot)$ is the number of divisors function which satisfies

$$d(n) \ll_{\varepsilon} n^{\varepsilon},$$

so

$$a(n_f) \ll_{\varepsilon} n_f^{\frac{2k-1}{2} + \varepsilon}.$$

By Lemma 3 we have

$$n_f \ll (4k^2N)^{\frac{3}{8}},$$

hence

$$C(f) \ll_{\varepsilon, k} N^{\frac{6k-3}{16} + \varepsilon}.$$

□

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